RANDOM DIRICHLET ENVIRONMENT VIEWED FROM THE PARTICLE IN DIMENSION $d \geq 3$

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Abstract: We consider random walks in random Dirichlet environment (RWDE) which is a special type of random walks in random environment where the exit probabilities at each site are i.i.d. Dirichlet random variables. On $\mathbb{Z}^d$, RWDE are parameterized by a $2d$-uplet of positive reals called weights. In this paper, we characterize for $d \geq 3$ the weights for which there exists an absolutely continuous invariant probability for the process viewed from the particle. We can deduce from this result and from [27] a complete description of the ballistic regime for $d \geq 3$.

1. Introduction

Multidimensional random walks in random environment have received a considerable attention in the last ten years. Some important progress has been made in the ballistic regime (after the seminal works [12, 33, 30, 31]) and for small perturbations of the simple random walk ([32, 1]). We refer to [35] for a detailled survey. Nevertheless, we are still far from a complete description and some basic questions are open such as the characterization of recurrence, ballisticity. The point of view of the environment viewed from the particle has been a powerful tool to investigate the random conductance model, it is a key ingredient in the proof of invariance principles ([14, 16, 28, 19]) but has had a rather little impact on the non-reversible model. The existence of an absolutely continuous invariant measure for the process viewed from the particle (the so called "equivalence of the static and dynamical point of view") is only known in a few cases: for dimension 1, cf Kesten [13] and Molchanov [20] p.273-274, in the case of balanced environment of Lawler, [17], for "non-nestling" RWRE in dimension $d \geq 4$ at low disorder, cf Bolthausen and Sznitman [5] and in a weaker form for ballistic RWRE (equivalence in half-space), cf [24, 25]. Note that invariance principles have nevertheless been obtained under special assumptions: under the ballistic assumption [25, 3] and for weak disorder in dimension $d \geq 3$, [32, 6].

Random walks in Dirichlet environment (RWDE) is a special case where at each site the environment is chosen according to a Dirichlet random variable. The annealed law of RWDE is the law of a directed edge reinforced random walk. While this model of environment is fully random (the support of the distribution on the environment is the space of weakly elliptic environment itself) its shows some surprising analytic simplifications (cf [26, 27, 9, 34]). In particular in [26] it is proved that RWDE are transient on transient graphs (cf [26] for a precise result). This result uses in a crucial way a property of statistical invariance by time reversal (cf lemma 1 of [26]).

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RWDE are parametrized by $2d$ reals called the weights (one for each direction in $\mathbb{Z}^d$) which govern the behavior of the walk. In this paper we characterize on $\mathbb{Z}^d$, $d \geq 3$, the weights for which there exists an invariant probability measure for the environment viewed from the particle, which is absolutely continuous with respect to the law of the environment. More precisely, it is shown that there is an absolutely continuous invariant probability exactly when the parameters are such that the time spent in finite size traps has finite expectation. Together with previous results on directional transience ([27]) it leads, using classical results on stationary ergodic sequences, to a complete description of the ballistic regimes for RWDE in dimension larger or equal to 3. Besides, we think that the proof of the existence of an absolutely continuous invariant distribution for the environment viewed from the particle could be a first step towards an implementation of the technics developed to prove functional central limit theorems (cf e.g. [15]).

2. Statement of the results

Let $(e_1, \dots, e_d)$ be the canonical base of $\mathbb{Z}^d$, and set $e_j = -e_{j-d}$, for $j = d+1, \dots, 2d$. The set $\{e_1, \dots, e_{2d}\}$ is the set of unit vectors of $\mathbb{Z}^d$. We denote by $\|z\| = \sum_{i=1}^d |z_i|$ the L_1-norm of $z \in \mathbb{Z}^d$. We write $x \sim y$ if $\|y - x\| = 1$. We consider elliptic random walks in random environment to nearest neighbors. We denote by Ω the set of environments

$$\Omega = \{\omega = (\omega(x,y))_{x\sim y} \in]0,1]^E, \;\text{ such that for all } x \in \mathbb{Z}^d,\; \sum_{i=1}^{2d} \omega(x, x+e_i) = 1\}.$$

An environment ω defines the transition probability of a Markov chain on $\mathbb{Z}^d$, and we denote by P_x^ω the law of this Markov chain starting from x:

$$P_x^\omega[X_{n+1} = y + e_i | X_n = y] = \omega(y, y+e_i).$$

The classical model of non-reversible random environment corresponds to the model where at each site $x \in \mathbb{Z}^d$ the environment $(\omega(x, x+e_i))_{i=1,\dots,2d}$ are chosen independently according to the same law. Random Dirichlet environment corresponds to the case where this law is a Dirichlet law. More precisely, we choose some positive weights $(\alpha_1, \dots, \alpha_{2d})$ and we define $\lambda = \lambda^{(\alpha)}$ as the Dirichlet law with parameters $(\alpha_1, \dots, \alpha_{2d})$. It means that $\lambda^{(\alpha)}$ is the law on the simplex

$$\{(x_1, \dots, x_{2d}) \in]0,1]^{2d}, \quad \sum_{i=1}^{2d} x_i = 1\} \tag{2.1}$$

with density

$$\frac{\Gamma(\sum_{i=1}^{2d} \alpha_i)}{\prod_{i=1}^{2d} \Gamma(\alpha_i)} \left(\prod_{i=1}^{2d} x_i^{\alpha_i - 1} \right) dx_1 \cdots dx_{2d-1}, \tag{2.2}$$

where Γ is the usual Gamma function $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$. (In the previous expression $dx_1 \cdots dx_{2d-1}$ represents the image of the Lebesgue measure on $\mathbb{R}^{2d-1}$ by the application $(x_1, \dots, x_{2d-1}) \to (x_1, \dots, x_{2d-1}, 1 - (x_1 + \cdots + x_{2d-1}))$. Obviously, the law does not depend on the specific role of x_{2d}.) We denote by $\mathbb{P}^{(\alpha)}$ the law obtained on Ω by picking at each site $x \in \mathbb{Z}^d$ the transition probabilities $(\omega(x, x+e_i))_{i=1,\dots,2d}$ independently according to $\lambda^{(\alpha)}$. We denote by $\mathbb{E}^{(\alpha)}$ the expectation with respect to $\mathbb{P}^{(\alpha)}$ and by $\mathbb{P}_x^{(\alpha)}[\cdot] = \mathbb{E}^{(\alpha)}[P_x^{(\omega)}(\cdot)]$ the annealed law of the process starting at x.

This type of environment plays a special role since the annealed law corresponds to a directed edge reinforced random walk with an affine reinforcement, i.e.

$$\mathbb{P}_x^{(\alpha)}[X_{n+1} = X_n + e_i | \sigma(X_k, k \le n)] = \frac{\alpha_i + N_i(X_n, n)}{\sum_{k=1}^{2d} \alpha_k + N_k(X_n, n)},$$

where $N_k(x, n)$ is the number of crossings of the directed edge $(x, x + e_k)$ up to time n (cf [21], [10]). When the weights are constant equal to α, the environment is isotropic: when α is large, the environment is close to the deterministic environment of the simple random walk, when α is small the environment is very disordered. The following parameter κ is important in the description of the RWDE

$$\kappa = 2\left(\sum_{i=1}^{2d} \alpha_i\right) - \max_{i=1,\dots,d} (\alpha_i + \alpha_{i+d}).$$

If $i_0 \in \{1, \dots, d\}$ realizes the maximum in the last term then κ is the sum of the weights of the edges exiting the set $\{0, e_{i_0}\}$ (or $\{0, -e_{i_0}\}$). The real κ must be understood as the strength of the trap $\{0, e_{i_0}\}$: indeed, if $\tilde{G}^\omega(0, 0)$ is the Green function at $(0, 0)$ of the Markov chain in environment ω killed at its exit time of the set $\{0, e_{i_0}\}$, then $\tilde{G}^\omega(0, 0)^s$ is integrable if and only if $s < \kappa$ ([34]). In [26] it has been proved for $d \ge 3$ that the same is true for the Green function $G(0, 0)$ on $\mathbb{Z}^d$ itself: it has integrable s-moment if and only if $s < \kappa$.

Denote by $(\tau_x)_{x\in\mathbb{Z}^d}$ the shift on the environment defined by

$$\tau_x\omega(y, z) = \omega(x + y, x + z).$$

Let X_n be the random walk in environment ω. The process viewed from the particle is the process on the state space Ω defined by

$$\overline{\omega}_n = \tau_{X_n}\omega.$$

Under $P_0^{\omega_0}$, $\omega_0 \in \Omega$ (resp. under $\mathbb{P}_0$) $\overline{\omega}_n$ is a Markov process on state space Ω with generator R given by

$$Rf(\omega) = \sum_{i=1}^{2d} \omega(0, e_i) f(\tau_{e_i}\omega),$$

for all bounded measurable function f on Ω, and with initial distribution δ_{ω_0} (resp. $\mathbb{P}$), cf e.g. [4]. Compared to the quenched process, the process viewed from the particle is Markovian. Since the state space is huge one needs, to take advantage if this point of view, to have the existence of an invariant probability measure, absolutely continuous with respect to the initial measure on the environment. The following theorem solves this problem in the special case of Dirichlet environment in dimension $d \ge 3$ and is the main result of the paper.

Theorem 1. *Let $d \ge 3$ and $\mathbb{P}^{(\alpha)}$ be the law of the Dirichlet environment with weights $(\alpha_1, \dots, \alpha_{2d})$. Let $\kappa > 0$ be defined by*

$$\kappa = 2\left(\sum_{i=1}^{2d} \alpha_i\right) - \max_{i=1,\dots,d} (\alpha_i + \alpha_{i+d}).$$

(i) If $\kappa > 1$ then there exists a unique probability distribution $\mathbb{Q}^{(\alpha)}$ on Ω absolutely continuous with respect to $\mathbb{P}^{(\alpha)}$ and invariant by the generator R. Moreover $\frac{d\mathbb{Q}^{(\alpha)}}{d\mathbb{P}^{(\alpha)}}$ is in $L_p(\mathbb{P}^{(\alpha)})$ for all $1 \le p < \kappa$.

(ii) If $\kappa \leq 1$, there does not exist any probability measure invariant by R and absolutely continuous with respect to the measure $\mathbb{P}^{(\alpha)}$.

We can deduce from this result and from [34], [27], a characterization of ballisticity for $d \geq 3$. Let d_α be the mean drift at first step:

$$d_\alpha = \mathbb{E}_0^{(\alpha)}(X_1) = \frac{1}{\sum_{i=1}^{2d} \alpha_i} \sum_{i=1}^{2d} \alpha_i e_i.$$

Theorem 2. *Let $d \geq 3$.*
i) (cf [34]) If $\kappa \leq 1$, then

$$\lim_{n\to\infty} \frac{X_n}{n} = 0, \quad \mathbb{P}_0^{(\alpha)} \ p.s.$$

ii) If $\kappa > 1$ and $d_\alpha = 0$ then

$$\lim_{n\to\infty} \frac{X_n}{n} = 0, \quad \mathbb{P}_0^{(\alpha)} \ p.s.$$

and for all $i = 1, \ldots, d$

$$\liminf X_n \cdot e_i = -\infty, \ \ \limsup X_n \cdot e_i = +\infty, \quad \mathbb{P}_0^{(\alpha)} \ p.s.$$

iii) If $\kappa > 1$ and $d_\alpha \neq 0$ then there exists $v \neq 0$ such that

$$\lim_{n\to\infty} \frac{X_n}{n} = v, \quad \mathbb{P}_0^{(\alpha)} \ p.s.$$

Moreover, for the integers $i \in \{1, \ldots, d\}$ such that $d_\alpha \cdot e_i \neq 0$ we have

$$(d_\alpha \cdot e_i)(v \cdot e_i) > 0.$$

For the integers $i \in \{1, \ldots, d\}$ such that $d_\alpha \cdot e_i = 0$

$$\liminf X_n \cdot e_i = -\infty, \ \ \limsup X_n \cdot e_i = +\infty, \quad \mathbb{P}_0^{(\alpha)} \ p.s.$$

Remark 1. *This answers in the case of RWDE for $d \geq 3$ the following question: is directional transience equivalent to ballisticity? The answer is formally "no" but morally "yes": indeed, it is proved in [27] that for all i such that $d_\alpha \cdot e_i \neq 0$, $X_n \cdot e_i$ is transient; hence, for $\kappa \leq 1$ directional transience and zero speed can coexist. But, it appears in the proof of [34] that the zero speed is due to finite size traps that come from the non-ellipticity of the environment. When $\kappa > 1$, the expected exit time of finite boxes is always finite (cf [34]) and in this case ii) and iii) indeed tell that directional transience is equivalent to ballisticity. For general RWRE (and for RWDE in dimension 2) this is an important unsolved question. Partial important results in this direction have been obtained by Sznitman in [30, 31] for general elliptic environment for $d \geq 2$.*

Remark 2. *A law of of large number (with eventually null velocity) has been proved for general (weakly) elliptic RWRE by Zerner, cf [37], using the technics of regeneration times developed by Sznitman and Zerner in [33]. Nevertheless, when the directional 0-1 law is not valid it is still not known wether there is a deterministic limiting velocity (this was solved for $d \geq 5$ by Berger, [2]).*

3. Proof of theorem 1 i)

Let us first recall a few definitions and give some notations. By a directed graph we mean a couple $G = (V, E)$ where V is a countable set of vertices and E the set of (directed) edges is a subset of $V \times V$. For simplicity, we do not allow multiple edges or loops (i.e. edges of the type (x, x)). We denote by $\underline{e}$, resp. $\overline{e}$, the tail and the head of an edge $e \in E$, so that $e = (\underline{e}, \overline{e})$. A directed path from a vertex x to a vertex y is a sequence $\sigma = (x_0 = x, \ldots, x_n = y)$ such that for all $i = 1, \ldots, n$, (x_{i-1}, x_i) is in E. The divergence operator is the function $\mathrm{div} : \mathbb{R}^E \mapsto \mathbb{R}^V$ defined for $\theta \in \mathbb{R}^E$ by

$$\forall x \in V, \quad \mathrm{div}(\theta)(x) = \sum_{e \in E,\, \underline{e}=x} \theta(e) - \sum_{e \in E,\, \overline{e}=x} \theta(e).$$

We consider $\mathbb{Z}^d$ as a directed graph: $G_{\mathbb{Z}^d} = (\mathbb{Z}^d, E)$ where the edges are the couple (x, y) such that $\|y - x\| = 1$. On E we consider the weights $(\alpha(e))_{e \in E}$ defined by

$$\forall x \in \mathbb{Z}^d,\ i = 1, \ldots, 2d, \quad \alpha((x, x + e_i)) = \alpha_i.$$

Hence, under $\mathbb{P}^{(\alpha)}$, at each site $x \in \mathbb{Z}^d$, the exit probabilities $(\omega(e))_{\underline{e}=x}$ are independent and distributed according to a Dirichlet law with parameters $(\alpha(e))_{\underline{e}=x}$.

When $N \in \mathbb{N}^*$, we denote by $T_N = (\mathbb{Z}/N\mathbb{Z})^d$ the d-dimensional torus of size N. We denote by $G_N = (T_N, E_N)$ the associated directed graph image of the graph $G = (\mathbb{Z}^d, E)$ by projection on the torus. We denote by $d(\cdot, \cdot)$ the shortest path distance on the torus. We write $x \sim y$ if $(x, y) \in E_N$. Let Ω_N be the space of (weakly) elliptic environments on T_N:

$$\Omega_N = \{\omega = (\omega(x, y))_{(x,y) \in E_N} \in]0, 1]^{E_N}, \ \text{ such that } \forall\, x \in T_N,\ \sum_{i=1}^{2d} \omega(x, x + e_i) = 1\}.$$

Ω_N is naturally identified with the space of the N-periodic environments on $\mathbb{Z}^d$. We denote by $\mathbb{P}_N^{(\alpha)}$ the Dirichlet law on the environment obtained by picking independently at each site $x \in T_N$ the exiting probabilities $(\omega(x, x + e_i))_{i=1,\ldots,2d}$ according to a Dirichlet law with parameters $(\alpha_i)_{i=1,\ldots,2d}$.

For ω in Ω_N we denote by $(\pi_N^\omega(x))_{x \in T_N}$ the invariant probability measure of the Markov chain on T_N in the environment ω (it is unique since the environments are elliptic). Let

$$f_N(\omega) = N^d \pi_N^\omega(0),$$

and

$$\mathbb{Q}_N^{(\alpha)} = f_N \cdot \mathbb{P}_N^{(\alpha)}.$$

Thanks to translation invariance, $\mathbb{Q}_N^{(\alpha)}$ is a probability measure on Ω_N. Theorem 1 is a consequence of the following lemma.

Lemma 1. *For all $p \in [1, \kappa[$*

$$\sup_{N \in \mathbb{N}} \|f_N\|_{L_p(\mathbb{P}_N^{(\alpha)})} < \infty.$$

Once this lemma is proved the proof of theorem 1 is routine argument (cf for example [4] page 18,19). Indeed, we consider $\mathbb{P}_N^{(\alpha)}$ and $\mathbb{Q}_N^{(\alpha)}$ as probability measures on N-periodic environments. Obviously, $\mathbb{P}_N^{(\alpha)}$ converges weakly to the probability measure $\mathbb{P}^{(\alpha)}$. By construction, $\mathbb{Q}_N^{(\alpha)}$ is an invariant probability measure for the process of the environment viewed from the particle. Since Ω is compact, we can find a

subsequence N_k such that $\mathbb{Q}^{(\alpha)}_{N_k}$ converges weakly to a probability measure $\mathbb{Q}^{(\alpha)}$ on Ω. The probability $\mathbb{Q}^{(\alpha)}$ is invariant for the process viewed from the particle, as a consequence of the invariance of $\mathbb{Q}^{(\alpha)}_N$. Let g be a continuous bounded function on Ω: we have for p such that $1<p<\kappa$ and $q=\frac{p}{p-1}$.

$$\begin{aligned}\left|\int g d\mathbb{Q}^{(\alpha)}\right| &= \left|\lim_{k\to\infty}\int g f_{N_k} d\mathbb{P}^{(\alpha)}_{N_k}\right|\\ &\leq \limsup_{k\to\infty}\left(\int |g|^q d\mathbb{P}^{(\alpha)}_{N_k}\right)^{\frac{1}{q}}\left(\int f_{N_k}^p d\mathbb{P}^{(\alpha)}_{N_k}\right)^{\frac{1}{p}}\\ &\leq c_p\|g\|_{L_q(\mathbb{P}^{(\alpha)})}\end{aligned}$$

where

$$c_p=\sup_{N\in\mathbb{N}}\|f_N\|_{L_p(\mathbb{P}^{(\alpha)}_N)}<\infty.$$

As a consequence $\mathbb{Q}^{(\alpha)}$ is absolutely continuous with respect to $\mathbb{P}^{(\alpha)}$ and

$$\left\|\frac{d\mathbb{Q}^{(\alpha)}}{d\mathbb{P}^{(\alpha)}}\right\|_{L_p(\mathbb{P}^{(\alpha)})}\leq c_p.$$

The uniqueness of $\mathbb{Q}^{(\alpha)}$ is classical and proved e.g. in [4] p. 11.

Proof. of lemma 1.

The proof is divided into three steps. The first step prepares the application of the property of "time reversal invariance" (lemma 1 of [26], or proposition 1 of [27]). The second step is a little trick to increase the weights in order to get the optimal exponent. The third step makes a crucial use of the "time-reversal invariance" and uses a lemma of the type "max-flow min-cut problem" proved in the next section.

Step 1: Let $(\omega_{x,y})_{x\sim y}$ be in Ω_N. The time-reversed environment is defined by

$$\check{w}_{x,y}=\pi_N^\omega(y)\omega_{y,x}\frac{1}{\pi_N^\omega(x)}$$

for x, y in T_N, $x\sim y$. At each point $x\in T_N$

$$\sum_{\underline{e}=x}\alpha(e)=\sum_{\overline{e}=x}\alpha(e)=\sum_{j=1}^{2d}\alpha_j,$$

It implies by lemma 1 of [26] that if $(\omega_{x,y})$ is distributed according to $\mathbb{P}^{(\alpha)}$, then $\check{w}$ is distributed according to $\mathbb{P}^{(\check{\alpha})}$ where

$$\forall (x,y)\in E_N,\quad \check{\alpha}_{(x,y)}=\alpha_{(y,x)}.$$

Let p be a real, $1<p<\kappa$.

$$\begin{aligned}(f_N)^p &= (N^d\pi_N^\omega(0))^p\\ &= \left(\frac{\pi_N^\omega(0)}{\frac{1}{N^d}\sum_{y\in T_N}\pi_N^\omega(y)}\right)^p\\ &\leq \prod_{y\in T_N}\left(\frac{\pi_N^\omega(0)}{\pi_N^\omega(y)}\right)^{p/N^d}\end{aligned}\tag{3.1}$$

where in the last inequality we used the arithmetico-geometric inequality. If $\theta : E_N \to \mathbb{R}_+$ we define $\check{\theta}$ by

$$\check{\theta}_{(x,y)} = \theta_{(y,x)}, \quad \forall x \sim y.$$

We clearly have

$$\frac{\check{\omega}^{\check{\theta}}}{\omega^{\theta}} = \pi_N^{\mathrm{div}(\theta)}, \tag{3.2}$$

where for γ and β two functions on E_N (resp. on T_N) we write γ^β for $\prod_{e\in E_N} \gamma(e)^{\beta(e)}$ (resp. $\prod_{x\in T_N} \gamma(x)^{\beta(x)}$). Hence, for all $\theta : E_N \mapsto \mathbb{R}_+$ such that

$$\mathrm{div}(\theta) = \frac{p}{N^d} \sum_{y\in T_N} (\delta_0 - \delta_y). \tag{3.3}$$

we have using (3.1), and (3.2)

$$f_N^p \quad \le \quad \frac{\check{\omega}^{\check{\theta}}}{\omega^{\theta}} \tag{3.4}$$

Step 2: Considering that $1 = \sum_{\|e\|=1} \omega(0, e)$, we have

$$1 = 1^\kappa \le (2d)^\kappa \sum_{i=1}^{2d} \omega(0, e_i)^\kappa.$$

Hence, we get

$$\mathbb{E}^{(\alpha)}\left(f_N^p\right) \quad \le \quad (2d)^\kappa \sum_{i=1}^{2d} \mathbb{E}^{(\alpha)}\left(\omega(0, e_i)^\kappa f_N^p\right)$$

Hence, we need now to prove that for all $i = 1, \dots, 2d$,

$$\sup_{N\in\mathbb{N}} \mathbb{E}^{(\alpha)}\left(\omega(0, e_i)^\kappa f_N^p\right) < \infty. \tag{3.5}$$

Considering (3.4), we need to prove that for all $i = 1, \dots, 2d$, we can find a sequence (θ_N), where $\theta_N : E_N \mapsto \mathbb{R}_+$ satisfies (3.3) for all N, such that

$$\sup_{N\in\mathbb{N}} \mathbb{E}^{(\alpha)}\left(\omega_{(0,e_i)}^\kappa \frac{\check{\omega}^{\check{\theta}_N}}{\omega^{\theta_N}}\right) < \infty. \tag{3.6}$$

Step 3: This is related to the max-flow min-cut problem (cf e.g. [18] section 3.1 or [11]). Let us first recall the notion of minimal cut-set sums on the graph $G_{\mathbb{Z}^d}$. A cut-set between $x \in \mathbb{Z}^d$ and ∞ is a subset S of E such that any infinite simple directed path (i.e. an infinite directed path that does not pass twice by the same vertex) starting from x must pass through one (directed) edge of S. A cut-set which is minimal for inclusion is necessarily of the form

$$S = \partial_+(K) = \{e \in E,\ \underline{e} \in K,\ \overline{e} \in K^c\}, \tag{3.7}$$

where K is a finite subset of $\mathbb{Z}^d$ containing x such that any $y \in K$ can be reached by a directed path in K starting at x. Let $(c_e)_{e\in E}$ be a set of non-negative reals called the capacities. The minimal cut-set sum between 0 and ∞ is defined as the value

$$m((c)) = \inf\{c(S), \quad S \text{ is a cut-set separating } 0 \text{ and } \infty\},$$

where $c(S) = \sum_{e\in S} c(e)$. Remark that the infimum can be taken only on minimal cut-set, i.e. cut-set of the form (3.7).

The proof uses the following lemma, whose proof is deferred to the next section since it is of a different nature.

Lemma 2. *Let C' and C'' be two reals such that $0 < C' < C'' < \infty$. There exists a constant $c_1 > 0$ and an integer $N_0 > 0$ depending only on C', C'', d, such that for all sequence $(c_e)_{e\in E}$ such that*

$$\forall e \in E, \quad C' < c_e < C'',$$

and for all integer $N > N_0$, there exists a function $\theta_N : E_N \mapsto \mathbb{R}_+$ such that

$$div(\theta_N) = m((c))\frac{1}{N^d}\sum_{x\in T_N}(\delta_0 - \delta_x), \tag{3.8}$$

$$\|\theta_N\|_2^2 = \sum_{e\in E_N}\theta_N(e)^2 < c_1$$

and such that

$$\theta_N(e) \le c(e), \quad \forall e \in E_N, \tag{3.9}$$

when we identify E_N with the edges of E such that $\underline{e} \in [-N/2, N/2[^d$.

The strategy now is to use this result to find a sequence (θ_N) which satisfies (3.6). Let $(\alpha^{(i)}(e))_{e\in E}$ be the weights obtained by increasing the weight α by κ on the edge $(0, e_i)$, and leaving the other values unchanged

$$\alpha^{(i)}(e) = \begin{cases} \alpha^{(i)}(e) = \alpha(e), & \text{if } e \neq (0, e_i), \\ \alpha^{(i)}((0, e_i)) = \alpha((0, e_i)) + \kappa = \alpha_i + \kappa. \end{cases}$$

Let us first remark that for all $i = 1, \dots, 2d$,

$$m((\alpha^{(i)})) \ge \kappa. \tag{3.10}$$

Take $i = 1, \dots, d$: if S contains the edge $(0, e_i)$ then $\alpha^{(i)}(S) \ge \alpha^{(i)}_{(0,e_i)} \ge \kappa$. Otherwise, for all $j = 1, \dots, d$, $j \neq i$, S must intersect the paths $(ke_j)_{k\in\mathbb{N}}$, $(-ke_j)_{k\in\mathbb{N}}$, $(0, e_i, (e_i + ke_j)_{k\in\mathbb{N}})$, $(0, e_i, (e_i - ke_j)_{k\in\mathbb{N}})$. These intersections are disjoints and it gives two edges with weights (α_j) and two edges with weights (α_{j+d}). Moreover, S must intersect the paths $(ke_i)_{k\in\mathbb{N}}$, $(-ke_i)_{k\in\mathbb{N}}$. It gives one edge with weight α_i and one with weight α_{i+d}. Hence,

$$\alpha^{(i)}(S) \ge 2\left(\sum_{j=1}^{2d}\alpha_j\right) - (\alpha_i + \alpha_{i+d}) \ge \kappa.$$

The same reasoning works for $i = d + 1, \dots, 2d$.

Let us now prove (3.6) for $i = 1$, the same reasoning works for all. We apply lemma 2 with $c(e) = \frac{p}{\kappa}\alpha^{(1)}(e)$. It gives for $N \ge N_0$ a function $\theta_N : E_N \mapsto \mathbb{R}_+$ which satisfies

$$\text{div}(\theta) = \frac{p}{N^d}\sum_{y\in T_N}(\delta_0 - \delta_y),$$

and $\theta_N(e) \le \frac{p}{\kappa}\alpha^{(1)}(e)$ and with bounded L_2 norm (indeed by (3.10), $m((\frac{p}{\kappa}\alpha^{(1)})) \ge \kappa$).

Let r, q be positive reals such that $\frac{1}{r} + \frac{1}{q} = 1$ and $pq < \kappa$. Using Hölder inequality and lemma 1 of [26] we get

$$\mathbb{E}^{(\alpha)}\left(\omega(0,e_1)^{\kappa}\frac{\check{\omega}^{\check{\theta}_N}}{\omega^{\theta_N}}\right) \leq \mathbb{E}^{(\alpha)}\left(\omega(0,e_1)^{q\kappa}\omega^{-q\theta_N}\right)^{1/q}\mathbb{E}^{(\alpha)}\left(\check{\omega}^{r\check{\theta}_N}\right)^{1/r}$$
$$= \mathbb{E}^{(\alpha)}\left(\omega(0,e_1)^{q\kappa}\omega^{-q\theta_N}\right)^{1/q}\mathbb{E}^{(\check{\alpha})}\left(\omega^{r\check{\theta}_N}\right)^{1/r}$$

We set $\alpha(x)=\sum_{\underline{e}=x}\alpha(e)$ and $\theta_N(x)=\sum_{\underline{e}=x}\theta_N(e)$. Remark that $\alpha(x)=\check{\alpha}(x)=\sum_{j=1}^{2d}\alpha_j$ for all $x\in T_N$. We set $\alpha_0=\sum_{j=1}^{2d}\alpha_j$. Simple computation gives

$$\mathbb{E}^{(\alpha)}\left(\omega(0,e_1)^{q\kappa}\omega^{-q\theta_N}\right)=$$
$$\left(\frac{\prod_{\substack{e\in E_N\\ e\neq(0,e_1)}}\Gamma(\alpha(e)-q\theta_N(e))}{\prod_{\substack{x\in T_N\\ x\neq 0}}\Gamma(\alpha_0-q\theta_N(x))}\right)\left(\frac{\Gamma(\alpha_1+q\kappa-q\theta_N((0,e_1)))}{\Gamma(\alpha_0+q\kappa-q\theta_N(0))}\right)\left(\frac{\prod_{x\in T_N}\Gamma(\alpha_0)}{\prod_{e\in E_N}\Gamma(\alpha(e))}\right)$$

Remark that all the terms are well-defined since $q\theta_N\leq\frac{pq}{\kappa}\alpha^{(1)}$ and $qp<\kappa$. We have the following inequalities.

$$\alpha_1(1-\frac{qp}{\kappa})\leq\alpha_1+q\kappa-q\theta_N((0,e_1))\leq\alpha_1+q\kappa,$$

and

$$\alpha_0(1-\frac{qp}{\kappa})\leq\alpha_0+q\kappa-q\theta_N(0)\leq\alpha_0+q\kappa,$$

which imply that

$$\mathbb{E}^{(\alpha)}\left(\omega(0,e_1)^{q\kappa}\omega^{-q\theta_N}\right)^{1/q}\leq A_1\left(\frac{\prod_{\substack{e\in E_N\\ e\neq(0,e_1)}}\Gamma(\alpha(e)-q\theta_N(e))}{\prod_{\substack{x\in T_N\\ x\neq 0}}\Gamma(\alpha_0-q\theta_N(x))}\right)^{1/q}\left(\frac{\prod_{\substack{x\in T_N\\ x\neq 0}}\Gamma(\alpha_0)}{\prod_{\substack{e\in E_N\\ e\neq(0,e_1)}}\Gamma(\alpha(e))}\right)^{1/q}$$

where

$$A_1=\left(\frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1)}\frac{\sup_{s\in[\alpha_1(1-\frac{qp}{\kappa}),\alpha_1+q\kappa]}\Gamma(s),}{\inf_{s\in[\alpha_0(1-\frac{qp}{\kappa}),\alpha_0+q\kappa]}\Gamma(s)}\right)^{1/q}.$$

Similarly, we get

$$\mathbb{E}^{(\check{\alpha})}\left(\omega^{r\check{\theta}_N}\right) = \left(\frac{\prod_{e\in E_N}\Gamma(\check{\alpha}(e)+r\check{\theta}_N(e))}{\prod_{x\in T_N}\Gamma(\check{\alpha}(x)+r\check{\theta}_N(x))}\right)\left(\frac{\prod_{x\in T_N}\Gamma(\check{\alpha}(x))}{\prod_{e\in E_N}\Gamma(\check{\alpha}(e))}\right)$$
$$= \left(\frac{\prod_{e\in E_N}\Gamma(\alpha(e)+r\theta_N(e))}{\prod_{x\in T_N}\Gamma(\alpha_0+r\check{\theta}_N(x))}\right)\left(\frac{\prod_{x\in T_N}\Gamma(\alpha_0)}{\prod_{e\in E_N}\Gamma(\alpha(e))}\right)$$

where in the last line we used that $\check{\alpha}((x,y))=\alpha((y,x))$ and $\check{\theta}((x,y))=\theta((y,x))$ and that $\check{\alpha}(x)=\sum_{\overline{e}=x}\alpha_e=\alpha(x)=\alpha_0$ for all x. Remark that $\check{\theta}(0)=\theta(0)-p$ and $\check{\theta}(x)=\theta(x)+\frac{p}{N^d}$ for $x\neq 0$, thanks to (3.3). We have the following inequalities

$$\alpha_1\leq\alpha((0,e_1))+r\theta_N((0,e_1))\leq\alpha_1(1+r)+r\kappa$$
$$\alpha_0\leq\alpha(0)+r\check{\theta}_N(0)\leq\alpha_0(1+r)+r\kappa.$$

This gives that

$$\mathbb{E}^{(\check{\alpha})}\left(\omega^{r\check{\theta}_N}\right)^{\frac{1}{r}} \leq A_2\left(\frac{\prod_{\substack{e\in E_N\\ e\neq(0,e_1)}}\Gamma(\alpha(e)+r\theta_N(e))}{\prod_{\substack{x\in T_N\\ x\neq 0}}\Gamma(\alpha_0+r\theta_N(x)+\frac{pr}{N^d})}\right)^{1/r}\left(\frac{\prod_{\substack{x\in T_N\\ x\neq 0}}\Gamma(\alpha_0)}{\prod_{\substack{e\in E_N\\ e\neq(0,e_1)}}\Gamma(\alpha(e))}\right)^{1/r}$$

where

$$A_2 = \left(\frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1)} \frac{\sup_{s\in[\alpha_1,\alpha_1(1+r)+r\kappa]}\Gamma(s)}{\inf_{s\in[\alpha_0,\alpha_0(1+r)+r\kappa]}\Gamma(s)} \right)^{1/r}.$$

Combining these inequalities it gives

$$\begin{aligned} &\mathbb{E}^{(\alpha)}\left(\omega(0,e_1)^\kappa \frac{\check{\omega}^{\check{\theta}_N}}{\omega^{\theta_N}} \right) \\ \leq\ & A_1A_2 \exp\left(\sum_{\substack{e\in E_N \\ e\neq(0,e_1)}} \nu(\alpha(e),\theta_N(e)) - \sum_{\substack{x\in T_N \\ x\neq 0}} \tilde{\nu}(\alpha_0,\theta_N(x)) \right), \end{aligned}$$

where

$$\nu(\alpha,u) = \frac{1}{r}\ln\Gamma(\alpha+ru) + \frac{1}{q}\ln\Gamma(\alpha-qu) - \ln\Gamma(\alpha).$$

and

$$\tilde{\nu}(\alpha,u) = \frac{1}{r}\ln\Gamma(\alpha+ru+\frac{pr}{N^d}) + \frac{1}{q}\ln\Gamma(\alpha-qu) - \ln\Gamma(\alpha).$$

Let $\underline{\alpha} = \min\alpha_i$, $\overline{\alpha} = \max\alpha_i$. By Taylor inequality and since $\underline{\alpha} \leq \alpha(e) \leq \overline{\alpha}$ for all $e \in E_N$, $q\theta_N(e) \leq \frac{qp}{\kappa}\alpha(e)$ for all $e \neq (0,e_1)$ and $qp < \kappa$, we can find a constant $c > 0$ such that for all $e \neq (0,e_1)$

$$|\nu(\alpha(e),\theta(e))| \leq c\theta(e)^2.$$

and for all $x \neq 0$

$$|\tilde{\nu}(\alpha_0,\theta(x))| \leq c(\theta(x)^2 + \frac{p}{N^d}).$$

Hence, we get a positive constante $C > 0$ independent of $N > N_0$ such that

$$\mathbb{E}^{(\alpha)}\left(\omega(0,e_1)^\kappa \frac{\check{\omega}^{\check{\theta}_N}}{\omega^{\theta_N}} \right) \leq \exp\left(C\left(\sum_{e\in E_N}\theta_N(e)^2 + \sum_{x\in T_N}\theta_N(x)^2 \right)\right),$$

Thus, (3.6) is thrue and this proves lemma 1. $\square$

4. Proof of lemma 2

The strategy is to apply the max-flow Min-cut theorem (cf [18] section 3.1 or [11]) to an appropriate choice of capacities on the graph G_N. We first need a generalized version of the max-flow min-cut theorem.

Proposition 1. *Let $G = (V,E)$ be a finite directed graph. Let $(c(e))_{e\in E}$ be a set of non-negative reals (called capacities). Let x_0 be a vertex and $(p_x)_{x\in V}$ be a set of non-negative reals. There exists a non-negative function $\theta : E \mapsto \mathbb{R}_+$ such that*

$$div(\theta) = \sum_{x\in V} p_x(\delta_{x_0} - \delta_x), \tag{4.1}$$

$$\forall e \in E, \quad \theta(e) \leq c(e), \tag{4.2}$$

if and only if for all subset $K \subset V$ containing x_0 we have

$$c(\partial_+ K) \geq \sum_{x\in K^c} p_x, \tag{4.3}$$

where $\partial_+ K = \{e \in E,\ \underline{e} \in K,\ \overline{e} \in K^c\}$ and $c(\partial_+ K) = \sum_{e\in\partial_+ K} c(e)$. The same is true if we restrict the condition (4.3) to the subsets K such that any $y \in K$ can be reached from 0 following a directed path in K.

Proof. If θ satisfies (4.1) and (4.2) then

$$\sum_{e,\ \underline{e}\in K, \overline{e}\in K^c} \theta(e) - \sum_{e,\ \overline{e}\in K, \underline{e}\in K^c} \theta(e) = \sum_{x\in K} \mathrm{div}(\theta)(x) = \sum_{x\in K^c} p_x.$$

It implies (4.3) by (4.2) and positivity of θ.

The reversed implication is an easy consequence of the classical max-flow min-cut theorem on finite directed graphs ([18] section 3.1 or [11]). Suppose now that (c) satisfies (4.3). Consider the new graph $\tilde{G} = (V \cup \delta, \tilde{E})$ defined by

$$\tilde{E} = E \cup \{(x, \delta),\ \ x \in V\}.$$

We consider the capacities $(\tilde{c}(e))_{e\in\tilde{E}}$ defined by $c(e) = \tilde{c}(e)$ for $e \in E$ and $c((x, \delta)) = p_x$. The strategy is to apply the max-flow min-cut theorem with capacities $\tilde{c}$ and with source x_0 and sink δ. Any minimal cutset between x_0 and δ in the graph $\tilde{G}$ is of the form $\partial_+^{\tilde{G}} K$ where $K \subset V$ is a subset containing x_0 but not δ and such that any point $y \in K$ can be reached from x_0 following a directed path in K. Remark that

$$\tilde{c}(\partial_+^{\tilde{G}} K) = c(\partial_+^{G} K) + \sum_{x\in K} p_x.$$

Hence, (4.3) implies

$$\tilde{c}(\partial_+^{\tilde{G}} K) \geq \sum_{x\in V} p_x.$$

Thus the max-flow min-cut theorem gives a flow $\tilde{\theta}$ on $\tilde{G}$ between x_0 and δ with strength $\sum_{x\in V} p_x$ and such that $\tilde{\theta} \leq \tilde{c}$. This necessarily implies that $\tilde{\theta}((x, \delta)) = p_x$. The function θ obtained by restriction of $\tilde{\theta}$ to E satisfies (4.2) and (4.1).

□

Lemma 3. *Let $d \geq 3$. There exists a positive constant $C_2 > 0$, such that for all $N > 1$, and all x, y in T_N there exists a unit flow θ from x to y (i.e. $\theta : E_N \to \mathbb{R}_+$ and $div(\theta) = \delta_x - \delta_y$) such that for all $z \in T_N$,*

$$\theta(z) = \sum_{\underline{e}=z} \theta(e) \leq 1 \wedge \left(C_2 \left(d(x, z)^{-(d-1)} + d(y, z)^{-(d-1)}\right)\right). \tag{4.4}$$

Proof. By translation and symmetry, we can consider only the case where $x = 0$ and $y \in [N/2, N[^d$ when T_N is identified with $[0, N[^d$. We construct a flow on $G_{\mathbb{Z}^d}$ supported by the set

$$D_y = [0, y_1] \times \cdots \times [0, y_d]$$

as an integral of sufficiently dispersed path flows. It thus induces by projection a flow on T_N with the same L_2 norm. Let us give some definitions. A sequence $\sigma = (x_0, \ldots, x_n)$ is a path from x to y in $\mathbb{Z}^d$ if $x_0 = x$, $x_n = y$ and $\|x_{i+1} - x_i\|_1 = 1$ for all $i = 1, \ldots, n$. We say that σ is a positive path if moreover $x_{i+1} - x_i \in \{e_1, \ldots, e_d\}$ for all $i = 1, \ldots, n$. To any path from x to y we can associate the unit flow from x to y defined by

$$\theta_\sigma = \sum_{i=1}^{n} \mathbb{1}_{(x_{i-1}, x_i)}.$$

For $u \in \mathbb{R}_+$, we define C_u by

$$C_u = \{z = (z_1, \ldots, z_d) \in \mathbb{R}_+^d, \ \sum_{i=1}^d z_i = u\}.$$

Clearly if $y \in \mathbb{N}^d$ and if $\sigma = (x_0 = 0, \ldots, x_n = y)$ is a positive path from 0 to y then $n = \|y\|_1$ and $x_k \in C_k$ for all $k = 0, \ldots, \|y\|$.

Set

$$\Delta_y = D_y \cap \{u = (u_1, \ldots, u_d) \in \mathbb{R}_+^d, \ \sum_{i=1}^d u_i = \frac{\|y\|_1}{2}\}.$$

For $u \in \Delta_y$, let L_u be the union of segments

$$L_u = [0, u] \cup [u, y].$$

We can consider L_u as the continuous path $l_u : [0, \|y\|] \mapsto D_y$ from 0 to y defined by

$$\{l_u(t)\} = L_u \cap C_t.$$

Remark that $u \in D_y$ implies that $l_u(t)$ is non-decreasing on each coordinates. There is a canonical way to associate with l_u a discrete positive path σ_u from 0 to y such that for all $k = 0, \ldots, \|y\|$

$$\|l_u(k) - \sigma_u(k)\| \le 2d. \tag{4.5}$$

Indeed, let $\tilde{l}_u(t)$ be defined by taking the integer part of each coordinate of $l_u(t)$. At jump times of $\tilde{l}_u(t)$ the coordinates increase at most by 1. We define $\sigma_u(k)$ as the positive path which follows the succesive jumps of $\tilde{l}_u(t)$: if at a time t there are jumps at several coordinates, we choose to increase first the coordinate on e_1, then on e_2... We have by construction $k - d \le \|\tilde{l}_u(k)\| \le k$, hence $\tilde{l}_u(k) \in \{\sigma_u(k-d), \ldots, \sigma_u(k)\}$, so $\|\sigma_u(k) - \tilde{l}_u(k)\| \le d$. Since $\|l_u(k) - \tilde{l}_u(k)\| \le d$ it gives (4.5). We then define

$$\theta_u = \theta_{\sigma_u},$$

and

$$\theta = \frac{1}{|\Delta_z|} \int_{\Delta_z} \theta_u du.$$

(where $|\Delta_z| = \int_{\Delta_z} du$) which is a unit flow from 0 to y. Clearly, $\theta(z) \le 1$ for all $z \in T_N$. For $k = 0, \ldots, \|y\|_1$ and $z \in \mathcal{H}_k$ we have

$$\theta(z) \le \frac{1}{|\Delta_z|} \int_{\Delta_z} \mathbb{1}_{\|l_u(k) - z\| \le 2d} du.$$

Hence, we have for k such that $1 < k \le \frac{\|y\|}{2}$

$$\theta(z) \le \frac{1}{|\Delta_z|} \int_{\Delta_z} \mathbb{1}_{\|u - z\frac{\|y\|}{2k}\| \le \frac{d\|y\|}{k}} du.$$

Since $y \in [N/2, N[^d$, there is a constant $C_2 > 0$ such that

$$\theta(z) \le C_2 k^{-(d-1)}.$$

Similarly, if $\|y\|/2 \le k < \|y\|$

$$\theta(z) \le C_2(\|y\| - k)^{-(d-1)}.$$

Moreover $\theta(z)$ is null on the complement of D_y. By projection on G_N it gives a function on E_N with the right properties. This proves lemma 3. $\square$

We are now ready to prove lemma 2. For all $y \in T_N$ we denote by $\theta_{0,y}$ a unit flow from 0 to y satisfying the conditions of lemma 3. We set

$$\tilde{\theta}_N = \frac{m(c)}{N^d} \sum_{y \in T_N} \theta_{0,y}.$$

The strategy is to apply proposition 1 to a set of capacities constructed from $\tilde{\theta}_N$ and c. Clearly,

$$\mathrm{div}(\tilde{\theta}_N) = \frac{m(c)}{N^d} \sum_{y \in T_N} (\delta_0 - \delta_y), \tag{4.6}$$

and by simple computation we get that

$$\tilde{\theta}_N(z) \leq C_2 m(c) \left(1 \wedge (d(0,z)^{-(d-1)}) + \frac{d2^d}{N^{d-1}} \right). \tag{4.7}$$

Hence,

$$\sum_{z \in T_N} \tilde{\theta}_N^2(z) \leq C_2^2 m^2(c)(d2^d)(1 + \sum_{k=1}^{\infty} k^{-(d-1)}) + C_2^2 m^2(c)(d2^d)^2 N^{-(d-2)},$$

and there is a constant $C_3 > 0$ depending solely on C', C'', d such that

$$\sum_{z \in T_N} \tilde{\theta}_N^2(z) \leq C_3, \quad \sum_{e \in E_N} \tilde{\theta}_N^2(e) \leq C_3.$$

By (4.6) we know that for all $K \subset T_N$ containing 0 we have

$$\sum_{e \in E_N,\, \underline{e} \in K,\, \overline{e} \in K^c} \tilde{\theta}_N(e) - \sum_{e \in E_N,\, \overline{e} \in K,\, \underline{e} \in K^c} \tilde{\theta}_N(e) = m(c) \frac{|K^c|}{N^d},$$

hence,

$$\tilde{\theta}_N(\partial_+ K) \geq m(c) \frac{|K^c|}{N^d}. \tag{4.8}$$

Let $(c(e))$ be such that $0 < C' < c(e) < C'' < \infty$. The strategy is to modify $\tilde{\theta}_N$ locally around 0 in order to make it lower or equal to c but large enough to be able to apply proposition 1. Let us fix some notations. For a positive integer r, $B_E(x_0, r)$ denotes the set of edges

$$B_E(x_0, r) = \{e \in E, \ \ \underline{e} \in B(x_0, r), \overline{e} \in B(x_0, r)\}.$$

and

$$\underline{B}_E(x_0, r) = \{e \in E, \ \ \underline{e} \in B(x_0, r)\}.$$

By (4.7), there exists η_0 and $\tilde{N}_0$ such that for all $N \geq \tilde{N}_0$ and $e \notin B_E(0, \eta_0)$ we have

$$|\tilde{\theta}_N(e)| \leq \frac{C'}{2}. \tag{4.9}$$

Choose now $\eta_1 > \eta_0$ such that

$$\eta_1 - \eta_0 \geq 2 \frac{m(c)}{C'}. \tag{4.10}$$

Finally we can find an integer $N_0 \geq \tilde{N}_0 \wedge (2\eta_1)$ large enough to satisfy

$$N_0^d \geq m(c) \frac{|B(0, \eta_1)|}{C'}. \tag{4.11}$$

We consider $(\tilde c_N(e))_{e\in E_N}$ defined by

$$\begin{cases} \tilde c_N(e)=c(e) & \text{if } \underline{e} \text{ or } \overline{e}\in B(0,\eta_1),\\ \tilde c_N(e)=\theta(e) & \text{otherwise .}\end{cases}$$

Remark that thanks to (4.9) for all $e\in E_N$, $\tilde c_N(e)\le c(e)$ when we identify E_N with the edges of E which starts in $[-N/2,N/2[^d$. In the rest of the proof we prove that for all $N\ge N_0$ and for all $K\subset T_N$ that contain 0 and which are such that any $y\in K$ can be reached from 0 following a directed path in K we have

$$\tilde c_N(\partial_+K)\ge m(c)\frac{|K^c|}{N^d}. \tag{4.12}$$

By application of proposition 1 it would give a flow θ_N which satisfies (3.8) and (3.9) and with a bounded L_2 norm, indeed,

$$\sum_{e\in E_N}\theta_N(e)^2\le C_3+|B_E(0,\eta_1)|(C'')^2.$$

We only need to check the inequality (4.12) for K such that K^c has a unique connected component. Indeed, if K^c has several connected components, say $R_1,\dots,R_k$, then

$$\partial_-R_i=\{e\in E_N,\ \ \overline{e}\in R_i,\ \underline{e}\in R_i^c\}=\{e\in E_N,\ \ \overline{e}\in R_i,\ \underline{e}\in K\}.$$

Hence, ∂_+K is the disjoint union of

$$\partial_+K=\sqcup_{i=1}^k\partial_-R_i.$$

Hence if we can prove (4.12) for $K_i=R_i^c$ we can prove it for K. Thus we assume moreover that K^c has a unique connected component in the graph G_N. There are four different cases.

- If $K\subset B(0,\eta_1)$ then
$$\tilde c(\partial_+K)=c(\partial_+K).$$
Moreover, viewed on $\mathbb{Z}^d$ (when T_N is identified with $[-N/2,N/2[)$ ∂_+K is a cut-set separating 0 from ∞ (indeed, $N\ge N_0\ge 2\eta_1$), thus
$$c(\partial_+K)\ge m(c)\ge m(c)\frac{|K^c|}{N^d}.$$
- If $B(0,\eta_0)\subset K$, by (4.8) and (4.9) then
$$\tilde c(\partial_+K)\ge\tilde\theta_N(\partial_+K)\ge m(c)\frac{|K^c|}{N^d}.$$
- If $K^c\subset B(0,\eta_1)$ then by (4.11)
$$\frac{|K^c|}{N^d}\le\frac{|B(0,\eta_1)|}{N_0^d}\le\frac{C'}{m(c)},$$
hence,
$$\tilde c(\partial_+K)=c(\partial_+K)\ge C'\ge m(c)\frac{|K^c|}{N^d}.$$
since $\partial K^c\neq\emptyset$.
- Otherwise K contains at least one point x_1 in $B(0,\eta_1)^c$ and K^c contains at least one point y_0 in $B(0,\eta_0)$ and one point y_1 in $B(0,\eta_1)^c$. Hence there is a path between y_0 and y_1 in K^c and a directed path between 0 and x_1 in K. Let $S(0,i)$ denotes the sphere with center 0 and radius i for the shortest path distance in G_N. It implies that we can find a sequence $z_{\eta_0},\dots,z_{\eta_1}$ such that $z_i\in K\cap S(0,i)$ and a sequence $z'_{\eta_0},\dots,z'_{\eta_1}$ such that $z'_i\in K^c\cap S(0,i)$. Since

there is a directed path on $S(0,i)$ between z_i and z_i' and a directed path in K between 0 and z_i, it implies that there exists at least $\eta_1 - \eta_0$ different edges in $\partial_+ K \cap B_E(0,\eta_1)$. Hence

$$\tilde{c}(\partial_+ K) \geq (\eta_1 - \eta_0)C' \geq m((c)).$$

This concludes the proof of (4.12) and of the lemma.

5. Proof of theorem 1 ii) and theorem 2

These results are based on classical results on ergodic stationary sequence, cf [8] page 343-344. Let us start with the following lemma.

Lemma 4. *Suppose that there exists an invariant probability measure $\mathbb{Q}^{(\alpha)}$, absolutely continuous with respect to $\mathbb{P}^{(\alpha)}$ and invariant for R. Then $\mathbb{Q}^{(\alpha)}$ is ergodic and equivalent to $\mathbb{P}^{(\alpha)}$. Let $(\Delta_i)_{i\geq 1}$ be the sequence*

$$\Delta_i = X_i - X_{i-1}.$$

Under the invariant annealed measure $\mathbb{Q}_0^{(\alpha)}(\cdot) = \mathbb{Q}^{(\alpha)}(P_0^\omega(\cdot))$ the sequence (Δ_i) is stationary and ergodic.

Proof. The first assertion on $\mathbb{Q}^{(\alpha)}$ is classical and proved e.g. in [4], chapter 2. Since $\mathbb{Q}^{(\alpha)}$ is an invariant probability measure for $\overline{\omega}_n$ it is clear that (Δ_i) is stationary. Let us prove it is ergodic. Suppose now that A is a measurable subset of $(\mathbb{Z}^d)^{\mathbb{N}}$ such that $\theta^{-1}(A) = A$ where θ is the time shift. Set

$$r(x,\omega) = P_x^\omega((\Delta_i) \in A), \quad r(\omega) = r(0,\omega).$$

For all environment ω

$$\lim_{n\to\infty} r(X_n,\omega) = \mathbb{1}_A((\Delta_i)), \quad P_x^\omega \text{ p.s.} \tag{5.1}$$

Indeed, we have

$$P_x^\omega((\Delta_i) \in A \mid \mathcal{F}_n) = P_x^\omega((\Delta_{i+n}) \in A \mid \mathcal{F}_n) = P_{X_n}^\omega((\Delta_i) \in A) = r(X_n,\omega)$$

where $\mathcal{F}_n = \sigma(X_0,\ldots,X_n)$. Hence, $r(X_n,\omega)$ is a bounded martingale and by the almost sure convergence theorem we get (5.1) since $\mathbb{1}_A((\Delta_i))$ is $\mathcal{F}_\infty$-measurable. Remark now that $r(X_n,\omega) = r(\overline{\omega}_n)$. Birkoff's ergodic theorem tells that for $\mathbb{Q}^{(\alpha)}$ almost all ω we have

$$\lim_{n\to\infty} \frac{1}{n}(r(X_0,\omega) + \cdots + r(X_{n-1},\omega)) = E^{\mathbb{Q}^{(\alpha)}}(r(\omega)), \quad P_0^\omega \text{ p.s.}$$

Comparing with (5.1) it implies that $E^{\mathbb{Q}^{(\alpha)}}(r(\omega)) \in \{0,1\}$. □

Proof. of theorem 1 ii). Suppose that there exists an invariant probability measure $\mathbb{Q}^{(\alpha)}$, absolutely continuous with respect to $\mathbb{P}^{(\alpha)}$ and invariant for R. Since (X_n) is $\mathbb{P}_0^{(\alpha)}$ p.s. (hence, $\mathbb{Q}_0^{(\alpha)}$ p.s.) transient ([26], theorem 1), it implies that

$$E^{\mathbb{Q}^{(\alpha)}}(P_0^\omega(H_0^+ = \infty)) > 0,$$

where H_0^+ is the first positive return time of X_n to 0. Let R_n be the number of point visited by (X_k) at time $n-1$

$$R_n = |\{X_k,\ k = 0,\ldots,n-1\}|.$$

Theorem 6.3.1 of [8] and lemma 4 tells that

$$\mathbb{P}_0^{(\alpha)} \text{ p.s.}, \quad \frac{R_n}{n} \to E^{\mathbb{Q}^{(\alpha)}}\left(P_0^\omega(H_0^+ = \infty)\right) > 0. \tag{5.2}$$

Let $i_0 \in \{1, \ldots, d\}$ be a direction which maximizes $\alpha_i + \alpha_{i+d}$. Theorem 3 of [34] tells that if $\kappa \le 1$ then the expected exit time under $\mathbb{P}_0^{(\alpha)}$ of the finite subset $\{0, e_{i_0}\}$ or $\{0, -e_{i_0}\}$ is infinite. By independence of the environment under $\mathbb{P}^{(\alpha)}$, we can easily get that $\frac{R_n}{n} \to 0$, $\mathbb{P}_0^{(\alpha)}$ p.s. This contradicts (5.2). □

Proof. of theorem 2. i) is proposition 11 of [34]. Under the annealed invariant law $\mathbb{Q}_0^{(\alpha)}$, (Δ_i) is a stationary ergodic sequence. Birkoff's ergodic theorem ([8], page 337) gives for free the law of large number

$$\mathbb{P}_0^{(\alpha)} \text{ p.s.}, \quad \frac{X_n}{n} \to E^{\mathbb{Q}^{(\alpha)}}\left(E_0^\omega(X_1)\right).$$

If $d_\alpha \cdot e_i = 0$ then by symmetry of the law of the environment it implies that $E^{\mathbb{Q}^{(\alpha)}}\left(E_0^\omega(X_1)\right) \cdot e_i = 0$, hence by theorem 6.3.2 of [8] we have ii) and the last assertion of iii).

For $l \in \mathbb{R}^d$ we set $A_l = \{X_n \cdot l \to \infty\}$. If $l \neq 0$ and if $\mathbb{P}_0^{(\alpha)}(A_l) > 0$ then Kalikow 0-1 law ([12], [36] proposition 3) tells that $\mathbb{P}_0^{(\alpha)}(A_l \cup A_{-l}) = 1$. Suppose now that $d_\alpha \cdot e_i > 0$ for $i \in 1, \ldots, 2d$. In [27] we proved that $\mathbb{P}_0^{(\alpha)}(A_{e_i}) > 0$, this implies that $X_n \cdot e_i$ visits 0 a finite number of times $\mathbb{Q}_0^{(\alpha)}$ p.s.. By theorem 6.3.2 of [8] and Birkoff's theorem it implies that

$$E^{\mathbb{Q}^{(\alpha)}}\left(E^\omega(X_1)\right) \cdot e_i > 0.$$

□

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